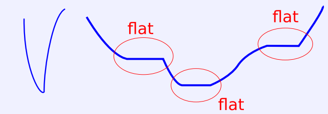
Relationships between different classes of functions

Quasiconvex

$\forall x_1, x_2 \in S, \lambda \in (0,1): \; f(\lambda x_1 + (1-\lambda)x_2) \leq \max\{f(x_1), f(x_2)\}$

"either monotonic, or unimodal"

"all level sets are convex"



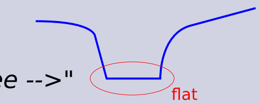
Semistrictly** quasiconvex

$\forall x_1, x_2 \in S: \; f(x_1) \neq f(x_2); \; \lambda \in (0,1): \; f(\lambda x_1 + (1-\lambda)x_2) < \max\{f(x_1), f(x_2)\}$

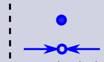
"no flatlands anywhere but optimum"

"local min = global min"

*"but global min is not necessarily unique****" -- see -->*



e.g.:



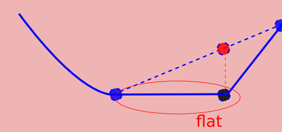
*semistrictly quasiconvex, **but not** quasiconvex*

Convex

$\forall x_1, x_2 \in S, \; \lambda \in (0,1): \; f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$

***Note:** a convex function is continuous over int S*

"any point on the line segment (red) is above the graph (black)"



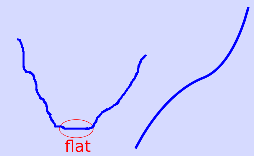
Pseudoconvex*

$\forall x_1, x_2 \in S, \; \nabla^T f(x_1)(x_2 - x_1) \geq 0 \to f(x_2) \geq f(x_1)$

$\forall x_1, x_2 \in S, \; f(x_2) < f(x_1) \to \nabla^T f(x_1)(x_2 - x_1) < 0$

"zero gradient implies global min (not necessarily unique)"

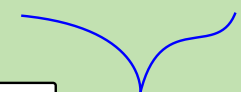
"minus gradient points towards min"



Strictly*** quasiconvex

$\forall x_1 \neq x_2 \in S, \lambda \in (0,1): \; f(\lambda x_1 + (1-\lambda)x_2) < \max\{f(x_1), f(x_2)\}$

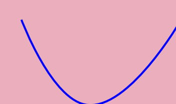
*"now local min = **unique******* global min (if exists)"*



Strictly convex

$\forall x_1, x_2 \in S, \lambda \in (0,1):$
$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2)$

Differentiable strictly convex



Strictly pseudoconvex*

$\forall x_1, x_2 \in S, x_1 \neq x_2, \nabla^T f(x_1)(x_2 - x_1) \geq 0 \to f(x_2) > f(x_1)$

$\forall x_1, x_2 \in S, x_1 \neq x_2, f(x_2) \leq f(x_1) \to \nabla^T f(x_1)(x_2 - x_1) < 0$

"now flatlands are prohibited"

(note that const. is not strictly pseudoconvex)

*"zero gradient implies **unique******* global min"*

Convex non-differentiable

Convex differentiable

*All **lower-semi-continuous** Sstrictly QC functions*
+ maybe some non-lower-semi-continuous Sstrictly quasiconvex functions

Notes: `S is assumed to be a convex set;`

`[!] phrases in "`*`quotes`*`" are rather mnemonic rules -- watch out for formal definitions and theorems!`

Main source: Bazaraa, Sherali, Shetty "Nonlinear Programming", 3rd ed.

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*     -- two equivalent formulations are provided
**    -- "strictly quasiconvex" in terms of Bazaraa et al.
***   -- "strongly convex" in terms of Bazaraa et al.
****  -- that is why it is rather "semi"-strictly
***** -- hence, "strictly"
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